Bosonic realizations of the color analogue of the Heisenberg Lie algebra

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November 12, 2003

Abstract. We describe realizations of the color analogue of the Heisenberg Lie algebra by power series in non-commuting indeterminates satisfying Heisenberg's canonical commutation relations of quantum mechanics. The obtained formulas are used to construct new operator representations of the color analogue of the Heisenberg Lie algebra. These representations are shown to be closely connected with some combinatorial identities and functional difference-differential interpolation formulae involving Euler, Bernoulli and Stirling numbers.

Mathematics Subject Classification 2000: Primary 17B75; Secondary 05A10,05A19, 16G99,16S32,34K99,81S05,81S99

1 Introduction

The main object studied in this paper is the unital associative algebra with three generators A_1, A_2 and A_3 satisfying defining commutation relations

$$\begin{aligned} A_1A_2 + A_2A_1 &= A_3, \\ A_1A_3 + A_3A_1 &= 0, \\ A_2A_3 + A_3A_2 &= 0. \end{aligned} \tag{1}$$

The main goal is to show how A_1, A_2 and A_3 can be expressed, using generators A, B of the Heisenberg algebra, obeying Heisenberg's canonical commutation relation

$$AB - BA = I. \tag{2}$$

The canonical representation of the commutation relation (2) is given by choosing A as usual differentiation operator and B as multiplication by x acting on differentiable functions of one real variable x, on polynomials in one variable, or on some other suitable linear space of functions invariant under these operators. In quantum mechanics, these operators, when considered on the Hilbert space of square integrable functions, are essentially the same as the canonical Heisenberg-Schrödinger observables of momentum and coordinate, differing just by a complex scaling factor. The Heisenberg canonical commutation relation (2) is also satisfied by the annihilation and creation operators in a quantum harmonic oscillator. Whenever A_1, A_2 and A_3 satisfy (1) and are interpreted as observables within some physical system, the problem we consider is that of realization of these observables within a physical system described by the Heisenberg-Schrödinger observables or by the quantum harmonic oscillator model. This point of view can be very valuable for physical applications as a step towards understanding bosonic realizations of fermionic, super-symmetric or color systems.

A complex associative algebra L with generators A_1, A_2, A_3 and defining relations (1) is called the graded analogue of the Heisenberg Lie algebra or, more precisely, of its universal enveloping algebra. The algebra L is a universal enveloping algebra of a three-dimensional $\mathbb{Z}_2^3$-graded generalized Lie algebra (see Appendix A). When anticommutators in the left-hand side of (1) are changed into commutators, we indeed have the relations between generators in the universal enveloping algebra of the Heisenberg Lie algebra.

Since the 1970's, generalized (color) Lie algebras have been an object of constant interest in both mathematics and physics [1, 2, 5, 6], [8]–[21], [25],

[27]–[34]. Description of representations of these algebras is an important and interesting general problem. It is well known that representations of three-dimensional Lie algebras play an important role in the representation theory of general Lie algebras and groups, both as test examples and building blocks. Similarly, one would expect the same to be true for three-dimensional color Lie algebras and superalgebras with respect to general color Lie algebras and superalgebras. The representations of non-isomorphic algebras have different structure. In [32, 34], three-dimensional color Lie algebras are classified in terms of their structure constants, that is in terms of commutation relations between generators. In [20], [13] and [31], quadratic central elements and involutions on these algebras are calculated. In [19] and [33], Hilbert space $*$-representations are described for the graded analogues of the Lie algebra $\mathfrak{sl}(2\,;\mathbb{C})$ and of the Lie algebra of the group of plane motions, two of the non-trivial algebras from the classification. The classification of $*$-representations in [19] and [33] is achieved, using the method of dynamical systems based on generalized Mackey imprimitivity systems.

The graded analogue of the Heisenberg Lie algebra defined by (1) is another important non-trivial algebra in the classification of three-dimensional color Lie algebras obtained in [32, 34]. In this article we look for representations of this algebra. Here, however, we approach representations in a totally different way than it was done in [19] and [33]. In this paper we are interested in describing those representations which can be obtained as power series in representations of Heisenberg's canonical commutation relations.

In Section 2 we show that, with a natural choice for A_1 as the first generator of the Heisenberg algebra corresponding to differentiation, there are no non-zero polynomials in Heisenberg generators which can be taken as A_2 and A_3 so that (1) is satisfied. This means, in particular, that when A_1 is the differentiation operator, A_2 and A_3 cannot be chosen as differential operators of finite order with polynomial coefficients. We prove however that it is possible for A_2 and A_3 to be power series in the Heisenberg generators with infinitely many non-zero terms, thus in particular making possible the operator representations by the differential operators of infinite order. In Theorem 2.1, we describe all such formal power series solutions A_2 and A_3 for the first two relations in (1). In Theorem 2.5, we present all formal power series solutions A_2 and A_3 satisfying all three relations in (1). It turns out that these solutions depend on the choice of two arbitrary odd power series, and thus on countably many arbitrary complex parameters. In other words, we get two mappings from the sequence space $\mathbb{C}^{\mathbb{N}}$ to formal power series

in Heisenberg generators, such that elements of their image spaces together with the first Heisenberg generator satisfy the commutation relations (1). In all these solutions one finds a special series, which turns out to be an abstract series generalization of the parity operator, playing an important role in quantum mechanics, quantum field theory and supersymmetry analysis.

By choosing various pairs of operators satisfying the Heisenberg canonical commutation relation (2) and substituting them into the power series obeying (1), one can find large classes of operator representations of the commutation relations (1). Section 3 is exclusively devoted to examples of such representations. Many of these representations, we believe, cannot be reached or classified using classical methods based on dynamical systems approach extending Mackey imprimitivity systems. We think that these operator representations might have significant physical applications. It would be of great interest to investigate spectral, structural and analytical properties of such representations on various spaces. It also turns out that for some of these representations, the commutation relations (1) lead to nontrivial functional differential-difference interpolation and combinatorial identities involving Euler, Bernoulli and Stirling numbers.

2 Bosonic power series realizations

Consider a set $\{A_1, A_2, A_3\}$ in some complex associative algebra with unit element I satisfying the following commutation relations

$$A_1A_2 + A_2A_1 = A_3, \quad A_1A_3 + A_3A_1 = 0, \quad A_2A_3 + A_3A_2 = 0.$$

It follows immediately that A_3^2 commutes with all three elements A_1, A_2 and A_3. Suppose there exists a non-zero constant α such that $A_3^2 = \alpha^2 I$. From the first relation we then obtain

$$A_1A_2A_3 + A_2A_1A_3 = A_3^2 = \alpha^2 I,$$

and using that $A_1A_3 = -A_3A_1$ by the second relation, we have

$$A_1(A_2A_3) - (A_2A_3)A_1 = \alpha^2 I.$$

Let $\hat{A}_2 = \alpha^{-1}A_2$ and $\hat{A}_3 = \alpha^{-1}A_3$. Then we can write

$$A_1(\hat{A}_2\hat{A}_3) - (\hat{A}_2\hat{A}_3)A_1 = I, \tag{3}$$

showing that A_1 and the combination $\hat{A}_2\hat{A}_3$ satisfy the Heisenberg canonical commutation relation. By the way this observation implies in particular that (1) together with $A_3^2 = \alpha^2 I,\ \alpha \neq 0$, cannot be satisfied by bounded operators on a Hilbert space or even generally by elements in any unital normed algebra, as this is also the case for the Heisenberg canonical commutation relation (2) by the famous Wintner-Wielandt result [24, 37, 36].

Assume that we consider A_1, A_2 and A_3 as elements of the Heisenberg algebra $\mathbb{C}\langle A, B\rangle/\langle AB - BA - I\rangle$. Then (3) suggests that a reasonable Ansatz is to put $A_1 = A$ and consider the other two generators A_2 and A_3 as polynomials in A and B having coefficients in $\mathbb{C}$. Suppose A_2 and A_3 are any polynomials in A and B. Due to the relation $AB = I + BA$, it is clear that A_2 and A_3 can be rewritten as a linear combination of monomials with no B to the right of A. When a polynomial (or a series) in A and B is written in such a way, we say that it is presented in its (B, A)-normal form. In the Heisenberg algebra, we know that the set of ordered monomials $\{B^j A^k \mid j, k \in \mathbb{N}\}$ is linearly independent. This fact allows one to reduce the problem of equality of two polynomials in A and B to checking whether they have the same coefficients when rewritten in (B, A)-normal form.

We begin with the following theorem, showing that if $A_1 = A$, then one is forced to work with series in A, B with infinitely many non-zero terms, in order to be able to find non-trivial realizations of the commutation relations

$$A_1A_2 + A_2A_1 = A_3, \quad A_1A_3 + A_3A_1 = 0$$

in terms of the Heisenberg generators A and B.

Theorem 2.1 *Assume A and B are two elements in an associative algebra over $\mathbb{C}$ with unit element I satisfying the Heisenberg canonical commutation relation $AB - BA = I$.*

(*a*) *Let $A_1 = A$, and both A_2 and A_3 be polynomials in A and B with complex coefficients. Then it follows that the commutation relations*

$$A_1A_2 + A_2A_1 = A_3, \quad A_1A_3 + A_3A_1 = 0$$

can only be satisfied if $A_2 = A_3 = 0$.

(*b*) *Let $A_1 = A$, and suppose A_2 and A_3 are formal power series in A and B in the (B, A)-normal form, i.e.*

$$A_2 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}B^jA^k, \quad A_3 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \tilde{a}_{jk}B^jA^k,$$

where the coefficients $a_{jk}, \tilde{a}_{jk} \in \mathbb{C}$*. Then* A_1*,* A_2 *and* A_3 *satisfy the commutation relations*

$$A_1A_2 + A_2A_1 = A_3, \quad A_1A_3 + A_3A_1 = 0$$

if and only if

$$\begin{aligned} A_2 &= T(A,B)V(A) + BT(A,B)W(A), \\ A_3 &= T(A,B)W(A), \end{aligned}$$

where $V(A)$ *and* $W(A)$ *are power series expressions in* A *with complex coefficients, and* $T(A,B)$ *is given by*

$$T(A,B) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} B^k A^k.$$

Proof. (a) Any polynomial in A and B can be written in the (B,A)-normal form, and hence, we can assume that

$$A_2 = \sum_{j=0}^{M} \sum_{k=0}^{N} a_{jk} B^j A^k \tag{4}$$

for some $M, N \in \mathbb{N}$ and $a_{jk} \in \mathbb{C}$. Eliminating A_3 by use of the commutation relations and using $A_1 = A$, we obtain

$$A_2A^2 + 2AA_2A + A^2A_2 = 0. \tag{5}$$

Introduce the notation

$$Q(A,B) = A_2A^2 + 2AA_2A + A^2A_2.$$

The reordering formula

$$AB^n = B^nA + nB^{n-1} \tag{6}$$

is valid for $n \geq 1$ and follows directly from the Heisenberg commutation relation (2) by induction on n [7, p. 21].
By repeated use of (6), we readily obtain for $n \geq 2$

$$\begin{aligned} A^2B^n &= A(AB^n) = A(B^nA + nB^{n-1}) = AB^nA + nAB^{n-1} \\ &= (B^nA + nB^{n-1})A + n(B^{n-1}A + (n-1)B^{n-2}) \\ &= B^nA^2 + nB^{n-1}A + nB^{n-1}A + (n-1)nB^{n-2} \\ &= B^nA^2 + 2nB^{n-1}A + (n-1)nB^{n-2}. \end{aligned} \tag{7}$$

Since the coefficients a_{jk} in (4) are allowed to be arbitrary complex numbers (including zero) one can, without loss of generality, put

$$A_2 = \sum_{j=0}^{N} \sum_{k=0}^{N} a_{jk} B^j A^k, \qquad N \geq 2,$$

and hence

$$A_2 A^2 = \sum_{j=0}^{N} \sum_{k=0}^{N} a_{jk} B^j A^{k+2}. \tag{8}$$

Applying repeatedly the reordering relations (6) and (7), we further obtain

$$\begin{aligned} A A_2 A &= \sum_{j=0}^{N} \sum_{k=0}^{N} a_{jk} A B^j A^{k+1} \\ &= \sum_{k=0}^{N} a_{0k} A^{k+2} + \sum_{j=1}^{N} \sum_{k=0}^{N} a_{jk} (B^j A + j B^{j-1}) A^{k+1} \\ &= \sum_{j=0}^{N} \sum_{k=0}^{N} a_{jk} B^j A^{k+2} + \sum_{j=1}^{N} \sum_{k=0}^{N} j a_{jk} B^{j-1} A^{k+1} \end{aligned} \tag{9}$$

and

$$\begin{aligned} A^2 A_2 &= \sum_{j=0}^{N} \sum_{k=0}^{N} a_{jk} A^2 B^j A^k = \sum_{k=0}^{N} a_{0k} A^{k+2} + \sum_{k=0}^{N} a_{1k} (B A^2 + 2A) A^k \\ &\quad + \sum_{j=2}^{N} \sum_{k=0}^{N} a_{jk} (B^j A^2 + 2j B^{j-1} A + (j-1) j B^{j-2}) A^k \\ &= \sum_{j=0}^{N} \sum_{k=0}^{N} a_{jk} B^j A^{k+2} + \sum_{j=1}^{N} \sum_{k=0}^{N} 2j a_{jk} B^{j-1} A^{k+1} \\ &\quad + \sum_{j=2}^{N} \sum_{k=0}^{N} (j-1) j a_{jk} B^{j-2} A^k. \end{aligned} \tag{10}$$

Inserting expressions (8), (9) and (10) into (5), we now have

$$
\begin{aligned}
Q(A,B) &= A_2A^2 + 2AA_2A + A^2A_2 \\
&= \sum_{j=0}^{N}\sum_{k=0}^{N} 4a_{jk}B^jA^{k+2} + \sum_{j=1}^{N}\sum_{k=0}^{N} 4ja_{jk}B^{j-1}A^{k+1} + \sum_{j=2}^{N}\sum_{k=0}^{N}(j-1)ja_{jk}B^{j-2}A^k \\
&= \sum_{j=0}^{N}\sum_{k=0}^{N} 4a_{jk}B^jA^{k+2} + \sum_{j=0}^{N-1}\sum_{k=-1}^{N-1} 4(j+1)a_{j+1,k+1}B^jA^{k+2} \\
&\quad + \sum_{j=0}^{N-2}\sum_{k=-2}^{N-2}(j+1)(j+2)a_{j+2,k+2}B^jA^{k+2} = 0. \qquad (11)
\end{aligned}
$$

Rearranging the sums, we arrive at

$$
\begin{aligned}
Q(A,B) = &\sum_{j=0}^{N-2}\sum_{k=0}^{N-2}[4a_{jk} + 4(j+1)a_{j+1,k+1} + (j+1)(j+2)a_{j+2,k+2}]B^jA^{k+2} \\
&+ \sum_{j=0}^{N-2}(j+1)(j+2)a_{j+2,0}B^j + \sum_{j=0}^{N-2}(j+1)(j+2)a_{j+2,1}B^jA \\
&+ \sum_{j=0}^{N-1}4(j+1)a_{j+1,0}B^jA + \sum_{j=0}^{N-1}4(j+1)a_{j+1,N}B^jA^{N+1} \\
&+ \sum_{k=0}^{N-2}4Na_{N,k+1}B^{N-1}A^{k+2} + \sum_{j=0}^{N}4a_{j,N-1}B^jA^{N+1} + \sum_{j=0}^{N}4a_{jN}B^jA^{N+2} \\
&+ \sum_{k=0}^{N-2}4a_{N-1,k}B^{N-1}A^{k+2} + \sum_{k=0}^{N-2}4a_{Nk}B^NA^{k+2}.
\end{aligned}
$$

The polynomial $Q(A,B)$ can now be presented in its (B,A)-normal form,

allowing us to express (5) as follows

$$
\begin{aligned}
Q(A,B) = {} & \sum_{j=0}^{N-2}\sum_{k=0}^{N-2}[4a_{jk} + 4(j+1)a_{j+1,k+1} + (j+1)(j+2)a_{j+2,k+2}]B^jA^{k+2} \\
& + \sum_{j=0}^{N-2}(j+1)(j+2)a_{j+2,0}B^j \\
& + \sum_{j=0}^{N-2}[4(j+1)a_{j+1,0} + (j+1)(j+2)a_{j+2,1}]B^jA + 4Na_{N0}B^{N-1}A \\
& + \sum_{j=0}^{N-1}4[a_{j,N-1} + (j+1)a_{j+1,N}]B^jA^{N+1} + 4a_{N,N-1}B^NA^{N+1} \\
& + \sum_{k=0}^{N-2}4(a_{N-1,k} + Na_{N,k+1})B^{N-1}A^{k+2} + \sum_{k=0}^{N-2}4a_{Nk}B^NA^{k+2} \\
& + \sum_{j=0}^{N}4a_{jN}B^jA^{N+2} = 0.
\end{aligned}
\tag{12}
$$

By linear independence of the set of ordered monomials $\{B^jA^k \mid j,k \in \mathbb{N}\}$, all coefficients must be equal to zero, giving rise to the following recurrence relation

$$
4a_{jk} + 4(j+1)a_{j+1,k+1} + (j+1)(j+2)a_{j+2,k+2} = 0, \tag{13}
$$

valid for $j,k \in \{0,\ldots,N-2\}$, together with the boundary conditions

$$
\begin{aligned}
(j+1)(j+2)a_{j+2,0} &= 0, & j &= 0,\ldots,N-2, \\
4(j+1)a_{j+1,0} + (j+1)(j+2)a_{j+2,1} &= 0, & j &= 0,\ldots,N-2, \\
Na_{N0} &= 0, \\
a_{j,N-1} + (j+1)a_{j+1,N} &= 0, & j &= 0,\ldots,N-1, \\
a_{N,N-1} &= 0, \\
a_{N-1,k} + Na_{N,k+1} &= 0, & k &= 0,\ldots,N-2, \\
a_{Nk} &= 0, & k &= 0,\ldots,N-2, \\
a_{jN} &= 0, & j &= 0,\ldots,N.
\end{aligned}
$$

It immediately follows, that this system of equations has the solution

$$
\begin{aligned}
a_{j0} &= 0, \qquad j = 2, \ldots, N, \\
a_{21} &= -2a_{10}, \quad a_{j1} = 0, \quad j = 3, \ldots, N, \\
a_{j,N-1} = a_{jN} &= 0, \qquad j = 0, \ldots, N, && (14) \\
a_{N-1,k} = a_{Nk} &= 0, \qquad k = 0, \ldots, N. && (15)
\end{aligned}
$$

Consider the square matrix (a_{ij}) of size $(N+1) \times (N+1)$. As expressed by the conditions (14)–(15), we see that the last two rows and last two columns consist merely of zeros. In view of relation (13), it clearly follows that (a_{ij}) must be the zero matrix, i.e. all coefficients $a_{jk} = 0$, showing that (5) cannot be satisfied by a non-zero polynomial expression in the form

$$A_2 = \sum_{j=0}^{M} \sum_{k=0}^{N} a_{jk} B^j A^k$$

for any $M, N \in \mathbb{N}$, proving part (a) of the theorem.

(b) Having now

$$A_2 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} B^j A^k,$$

it follows immediately that (11) will be replaced by

$$
\begin{aligned}
&A_2 A^2 + 2AA_2A + A^2A_2 \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} [4a_{jk} + 4(j+1)a_{j+1,k+1} + (j+1)(j+2)a_{j+2,k+2}] B^j A^{k+2} \\
&\quad + \sum_{j=0}^{\infty} (j+1)(j+2)a_{j+2,0} B^j + \sum_{j=0}^{\infty} (j+1)[4a_{j+1,0} + (j+2)a_{j+2,1}] B^j A = 0.
\end{aligned}
$$

We still have the recurrence relation

$$4a_{jk} + 4(j+1)a_{j+1,k+1} + (j+1)(j+2)a_{j+2,k+2} = 0, \tag{16}$$

now valid for $j, k \in \mathbb{N}$, together with the conditions

$$
\begin{aligned}
(j+1)(j+2)a_{j+2,0} &= 0, \qquad j \in \mathbb{N}, \\
(j+1)[4a_{j+1,0} + (j+2)a_{j+2,1}] &= 0, \qquad j \in \mathbb{N}
\end{aligned}
$$

or more explicitly

$$a_{j0} = 0, \qquad j = 2, 3, \ldots, \tag{17}$$
$$a_{21} = -2a_{10}, \tag{18}$$
$$a_{j1} = 0, \qquad j = 3, 4, \ldots. \tag{19}$$

As a consequence of (16), (17) and (19) we have

$$a_{i+2+j,j} = 0, \quad i, j \in \mathbb{N}. \tag{20}$$

The elements $a_{j+1,j}$, $j \in \mathbb{N}$, can be computed from (16) and (18), and will be treated separately below. In the relation (16), we now put $k = j + l$, obtaining

$$4a_{j,j+l} + 4(j+1)a_{j+1,j+l+1} + (j+1)(j+2)a_{j+2,j+l+2} = 0, \tag{21}$$

where $j, l \in \mathbb{N}$. The substitution $b_j^l = a_{j,j+l}$ brings (21) to the form

$$4b_j^l + 4(j+1)b_{j+1}^l + (j+1)(j+2)b_{j+2}^l = 0. \tag{22}$$

Suppressing the superscript l for a moment, we look for a general solution $(b_i)_{i=0}^{\infty}$ to the difference equation

$$4b_j + 4(j+1)b_{j+1} + (j+1)(j+2)b_{j+2} = 0. \tag{23}$$

In order to solve (23), we introduce the generating function

$$y(t) = \sum_{j=0}^{\infty} b_j t^j. \tag{24}$$

Successive differentiation of the series expression yields

$$y'(t) = \sum_{j=0}^{\infty} (j+1)b_{j+1}t^j, \tag{25}$$
$$y''(t) = \sum_{j=0}^{\infty} (j+1)(j+2)b_{j+2}t^j. \tag{26}$$

(23), together with the expressions (24), (25) and (26), gives rise to the differential equation

$$y''(t) + 4y'(t) + 4y(t) = 0, \tag{27}$$

where y has to satisfy the initial conditions

$$y(0) = b_0, \quad y'(0) = b_1. \tag{28}$$

The characteristic equation $(r+2)^2 = 0$ has the root $r = -2$ of multiplicity 2, so the general solution has the form

$$y(t) = (C_1 + C_2 t)e^{-2t}.$$

By conditions (28)

$$C_1 = b_0, \quad C_2 = b_1 + 2b_0,$$

so the solution is

$$y(t) = (b_0 + b_1 t + 2b_0 t)e^{-2t}.$$

Expanding the exponential function yields

$$\begin{aligned} y(t) = \sum_{j=0}^{\infty} b_j t^j &= b_0 \sum_{j=0}^{\infty} \frac{(-2)^j}{j!} t^j + (2b_0 + b_1) \sum_{j=0}^{\infty} \frac{(-2)^j}{j!} t^{j+1} \\ &= b_0 + \sum_{j=0}^{\infty} \left[b_0 \frac{(-2)^{j+1}}{(j+1)!} + 2b_0 \frac{(-2)^j}{j!} + b_1 \frac{(-2)^j}{j!} \right] t^{j+1}. \end{aligned}$$

After identification of the coefficients we have

$$b_{j+1} = \frac{(-2)^j}{j!} \left(b_1 + \frac{2j}{j+1} b_0 \right), \qquad j = 1, 2, \ldots.$$

Going back to the earlier notation this means

$$b_{j+1}^l = \frac{(-2)^j}{j!} \left(b_1^l + \frac{2j}{j+1} b_0^l \right), \qquad j = 1, 2, \ldots, \quad l = 0, 1, \ldots,$$

and hence,

$$a_{j+1,j+l+1} = \frac{(-2)^j}{j!} \left(a_{1,l+1} + \frac{2j}{j+1} a_{0l} \right), \quad j = 1, 2, \ldots, \quad l = 0, 1, \ldots. \tag{29}$$

If, in the formula (29) we put $j = 1,\ l = -1$, then

$$a_{21} = -2(a_{10} + a_{0,-1}) = -2a_{10}$$

by introducing an auxiliary coefficient $a_{0,-1} = 0$. The solution to the problem (16), (17), (18) and (19) can now be written

$$a_{j+1,j+l} = \frac{(-2)^j}{j!}\left(a_{1l} + \frac{2j}{j+1}a_{0,l-1}\right), \tag{30}$$

$$a_{j+l+2,l} = 0, \tag{31}$$

where we have $j, l \in \mathbb{N}$. By virtue of (31), we may write

$$\begin{aligned} A_2 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}B^j A^k &= \sum_{k=0}^{\infty} a_{0k}A^k + \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} a_{jk}B^j A^k \\ &= \sum_{k=0}^{\infty} a_{0k}A^k + \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} a_{j+1,j+l}B^{j+1}A^{j+l} \end{aligned} \tag{32}$$

and hence, directly by reordering formula (6)

$$\begin{aligned} A_3 = AA_2 + A_2A &= \sum_{k=0}^{\infty} a_{0k}A^{k+1} + \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} a_{j+1,j+l}AB^{j+1}A^{j+l} \\ &+ \sum_{k=0}^{\infty} a_{0k}A^{k+1} + \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} a_{j+1,j+l}B^{j+1}A^{j+l+1} \\ &= 2A_2A + \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}(j+1)a_{j+1,j+l}B^jA^{j+l}. \end{aligned} \tag{33}$$

Let $c_0 = 0$ and assume $(c_i)_{i=1}^{\infty}$ and $(d_i)_{i=0}^{\infty}$ are arbitrary sequences of complex numbers. For $l = 0, 1, 2, \dots$ we put

$$a_{0,l-1} = c_l, \quad a_{1l} = d_l \tag{34}$$

and have then for $j, l \in \mathbb{N}$

$$a_{j+1,j+l} = \frac{(-2)^j}{j!}\left(d_l + \frac{2j}{j+1}c_l\right). \tag{35}$$

Inserting now the expressions (34) and (35) into (32) and (33), yields

$$\begin{aligned}
A_2 &= \sum_{k=0}^{\infty} c_{k+1} A^k + \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-2)^j}{j!}\left(d_l + \frac{2j}{j+1}c_l\right) B^{j+1}A^{j+l} \\
&= \sum_{k=0}^{\infty} c_{k+1} A^k + \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-2)^{j+1}}{(j+1)!}[-(j+1)d_l/2 - jc_l] B^{j+1}A^{j+l} \\
&= \sum_{\substack{k,l=0\\k+l\neq 0}}^{\infty} \frac{(-2)^k}{k!}[(1-k)c_l - kd_l/2] B^k A^{k+l-1},
\end{aligned}$$

$$\begin{aligned}
A_3 &= 2A_2A + \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-2)^j}{j!}(j+1)\left(d_l + \frac{2j}{j+1}c_l\right) B^j A^{j+l} \\
&= 2A_2A + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-2)^k}{k!}[(k+1)d_l + 2kc_l] B^k A^{k+l} \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-2)^k}{k!}[2(1-k)c_l - kd_l + (k+1)d_l + 2kc_l] B^k A^{k+l} \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-2)^k}{k!}(2c_l + d_l) B^k A^{k+l}.
\end{aligned}$$

For $l \in \mathbb{N}$, we introduce the coefficients $w_l = 2c_l + d_l$, allowing us to write

$$\begin{aligned}
A_2 &= \sum_{\substack{k,l=0\\k+l\neq 0}}^{\infty} \frac{(-2)^k}{k!}(c_l - \tfrac{1}{2}kw_l) B^k A^{k+l-1}, \\
A_3 &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-2)^k}{k!} w_l B^k A^{k+l},
\end{aligned}$$

where the coefficients c_l, w_l are arbitrary complex constants except for c_0, since by definition (34) we have $c_0 = 0$. Alternatively, we can express A_2 and

A_3 by separating the summations as follows

$$\begin{aligned}
A_2 &= \sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\frac{(-2)^k}{k!}c_l B^k A^{k+l-1} + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-2)^{k-1}}{(k-1)!}w_l B^k A^{k-1+l} \\
&= \sum_{k=0}^{\infty}\frac{(-2)^k}{k!}B^k A^k \sum_{l=1}^{\infty} c_l A^{l-1} + B\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}B^k A^k \sum_{l=0}^{\infty} w_l A^l, \qquad (36) \\
A_3 &= \sum_{k=0}^{\infty}\frac{(-2)^k}{k!}B^k A^k \sum_{l=0}^{\infty} w_l A^l. \qquad (37)
\end{aligned}$$

■

In the following lemma, we formulate some basic rules, that will be used frequently below in proving a corollary to Theorem 2.1 and for the proof of our main theorem (Theorem 2.5).

Lemma 2.2 *Assume A and B are two elements in some complex associative algebra with unity I satisfying the Heisenberg canonical commutation relation $AB - BA = I$. Let $f(A)$ and $g(B)$ be formal power series in A and B, respectively. Denote by $f'(A)$ and $g'(B)$ their formal derivatives, obtained by term-wise differentiation of the series expressions, and let $T(A,B)$ be given by*

$$T(A,B) = \sum_{k=0}^{\infty}\frac{(-2)^k}{k!}B^k A^k. \qquad (38)$$

Then the following relations hold true

(*a*) $T(A,B)^2 = T(A,B)T(A,B) = I$,

(*b*) $f(A)B = Bf(A) + f'(A), \quad Ag(B) = g(B)A + g'(B)$,

(*c*) $AT(A,B) = -T(A,B)A, \quad T(A,B)B = -BT(A,B)$,

(*d*) $f(A)T(A,B) = T(A,B)f(-A), \quad T(A,B)g(B) = g(-B)T(A,B)$.

Proof. The reordering relation [7, Cor 2.4 p. 24]

$$A^i B^j = \sum_{\nu=0}^{\min(i,j)} \nu!\binom{i}{\nu}\binom{j}{\nu} B^{j-\nu}A^{i-\nu} \qquad (39)$$

is valid for all non-negative i and j, as long as A and B satisfy the Heisenberg canonical commutation relation (2).
(a) Using the definition (38) and relation (39), we have

$$\begin{aligned}
T(A,B)T(A,B) &= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} B^k A^k \sum_{m=0}^{\infty} \frac{(-2)^m}{m!} B^m A^m \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-2)^{k+m}}{k!\ m!} B^k A^k B^m A^m \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\nu=0}^{\min(k,m)} \frac{(-2)^{k+m}}{k!\ m!} \nu! \binom{k}{\nu} \binom{m}{\nu} B^{k+m-\nu} A^{k+m-\nu} \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\nu=0}^{\min(k,m)} \frac{(-2)^{k+m}}{k!\ (m-\nu)!} \binom{k}{\nu} B^{k+m-\nu} A^{k+m-\nu}.
\end{aligned}$$

Introducing a new summation index $r = k+m-\nu$, this can be expressed as

$$T(A,B)T(A,B) = \sum_{r=0}^{\infty} d_r B^r A^r,$$

where

$$\begin{aligned}
d_r &= \sum_{k=0}^{r} \sum_{m=r-k}^{r} \frac{(-2)^{k+m}}{k!\ (r-k)!} \binom{k}{k+m-r} \\
&= (-2)^r \sum_{k=0}^{r} \frac{1}{k!\ (r-k)!} \sum_{\nu=0}^{k} (-2)^{\nu} \binom{k}{\nu} \\
&= \frac{(-2)^r}{r!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} = \frac{(-2)^r}{r!} \delta_{r0} = \delta_{r0}.
\end{aligned}$$

Hence,

$$T(A,B)T(A,B) = \sum_{r=0}^{\infty} d_r B^r A^r = \sum_{r=0}^{\infty} \delta_{r0} B^r A^r = I.$$

(b) From the reordering relation (39), we obtain $A^n B = BA^n + nA^{n-1}$, valid for all $n \geq 1$. Thus, the first relation in (b) is true for $f(A) = A^n$, $n \in \mathbb{N}$, and so clearly it also holds for every formal power series in A. Moreover, for $n \geq 1$ we have by (6) or from relation (39) that $AB^n = B^n A + nB^{n-1}$, showing that

the second equation (b) holds for $g(B) = B^n$, $n \in \mathbb{N}$, and therefore it is also valid for any formal power series in B.
(c) By use of the second relation in (b), we can write

$$\begin{aligned}
AT(A,B) &= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} AB^k A^k = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} (B^k A) A^k + \sum_{k=1}^{\infty} \frac{(-2)^k}{k!} (kB^{k-1}) A^k \\
&= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} B^k A^k A - 2 \sum_{k=1}^{\infty} \frac{(-2)^{k-1}}{(k-1)!} B^{k-1} A^k \\
&= T(A,B)A - 2T(A,B)A = -T(A,B)A.
\end{aligned}$$

Applying the first relation in (b), it follows that

$$\begin{aligned}
T(A,B)B &= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} B^k A^k B = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} B^k (BA^k) + \sum_{k=1}^{\infty} \frac{(-2)^k}{k!} B^k (kA^{k-1}) \\
&= B \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} B^k A^k - 2B \sum_{k=1}^{\infty} \frac{(-2)^{k-1}}{(k-1)!} B^{k-1} A^{k-1} \\
&= BT(A,B) - 2BT(A,B) = -BT(A,B).
\end{aligned}$$

(d) Using the first of relations (c), we have by induction on n that

$$A^n T(A,B) = T(A,B)(-A)^n$$

for all $n \geq 0$, and hence, the first relation follows for any power series $f(A)$. From the second relation in (c), we have by induction on n that

$$T(A,B)B^n = (-B)^n T(A,B)$$

for all $n \geq 0$, and hence, the second relation holds for an arbitrary power series $g(B)$. ■

Remark 2.3 The series $T(A,B)$ can be seen as an abstract generalization of the parity operator $f(x) \mapsto f(-x)$. The usual parity operator is obtained in the special case of canonical representation of the Heisenberg relation (2) when $A = \partial_x : f(x) \mapsto f'(x)$ is differentiation and $B = M_x : f(x) \mapsto xf(x)$ is multiplication operator acting on functions on $\mathbb{R}$. This is proved in the beginning of section 3.

In view of the discussion in the beginning of Section 2, it is of interest to have a closer look at the compositions $A_2 A_3$ and A_3^2.

Corollary 2.4 *Assume A and B are two elements in some complex associative algebra with unity I satisfying the Heisenberg canonical commutation relation $AB - BA = I$. Let $A_1 = A$, and suppose A_2 and A_3 are formal power series in A and B in the (B, A)-normal form, i.e.*

$$A_2 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk} B^j A^k, \quad A_3 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \tilde{a}_{jk} B^j A^k,$$

where the coefficients $a_{jk}, \tilde{a}_{jk} \in \mathbb{C}$. If A_1, A_2 and A_3 satisfy the commutation relations

$$A_1A_2 + A_2A_1 = A_3, \quad A_1A_3 + A_3A_1 = 0$$

then it follows that

$$\begin{aligned} A_2A_3 &= V(-A)W(A) + BW(-A)W(A), \\ A_3^2 &= W(-A)W(A), \end{aligned}$$

where $V(A)$ and $W(A)$ are formal power series expressions in A with complex coefficients.

Proof. The general solution to the problem according to Theorem 2.1 can be expressed as

$$A_2 = T(A, B)V(A) + BT(A, B)W(A), \tag{40}$$
$$A_3 = T(A, B)W(A), \tag{41}$$

where $V(A)$ and $W(A)$ are arbitrary formal power series with coefficients in $\mathbb{C}$. Direct substitution of these expressions and application of the rules in Lemma 2.2, yields

$$\begin{aligned} A_2A_3 &= T(A, B)V(A)T(A, B)W(A) + BT(A, B)W(A)T(A, B)W(A) \\ &= T(A, B)T(A, B)V(-A)W(A) + BT(A, B)T(A, B)W(-A)W(A) \\ &= V(-A)W(A) + BW(-A)W(A), \\ A_3A_3 &= T(A, B)W(A)T(A, B)W(A) = T(A, B)T(A, B)W(-A)W(A) \\ &= W(-A)W(A). \end{aligned}$$

■

In Theorem 2.1 we stated the general (B, A)-normal form of the power series A_2 and A_3 satisfying the first two relations in (1). We now turn our

attention to investigating the possibility of satisfying also the third relation, namely

$$A_2A_3 + A_3A_2 = 0. \tag{42}$$

In the following theorem, being the main result of this article, we give the general solution to the problem with all three relations (1). In the formulation the exponential generating function $E(t)$ of the so-called Euler numbers is used. For basic facts about the Euler polynomials and Euler numbers, we refer to Appendix B.

Theorem 2.5 *Suppose A and B are two elements in a unital associative algebra over $\mathbb{C}$ with unity I satisfying the Heisenberg canonical commutation relation $AB - BA = I$. Put $A_1 = A$ and let A_2, A_3 and $T(A,B)$ be formal power series in A and B in the (B,A)-normal form given as*

$$A_2 = \sum_{j,k=0}^{\infty} a_{jk}B^jA^k, \quad A_3 = \sum_{j,k=0}^{\infty} \tilde{a}_{jk}B^jA^k, \quad T(A,B) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!}B^kA^k,$$

where the coefficients $a_{jk}, \tilde{a}_{jk} \in \mathbb{C}$. If A_1, A_2 and A_3 satisfy the commutation relations

$$A_1A_2 + A_2A_1 = A_3, \quad A_1A_3 + A_3A_1 = 0, \quad A_2A_3 + A_3A_2 = 0,$$

then either $A_3 = 0$ and $A_2 = T(A,B)V(A)$, where $V(A)$ is a formal power series in A with complex coefficients, or

$$\begin{aligned} A_2 &= c\,T(A,B)E(\varphi(A))[e^{\varphi(A)}\psi(A) - \tfrac{1}{2}\varphi'(A)] + c\,BT(A,B)e^{\varphi(A)}, \\ A_3 &= c\,T(A,B)e^{\varphi(A)}, \end{aligned}$$

where c is a non-zero complex constant, $E(t)$ is the exponential generating function of the Euler numbers E_k given by

$$E(t) = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!}t^{2n},$$

and both $\varphi(A)$ and $\psi(A)$ are odd formal power series expressions in A with complex coefficients.

Proof. By Theorem 2.1 we have, when considering only the first two relations, a general solution of the form given by (40) and (41). In the

present case, A_2 and A_3 are supposed to satisfy the additional condition $A_2A_3 + A_3A_2 = 0$. By Corollary 2.4 we have

$$A_2A_3 = V(-A)W(A) + BW(-A)W(A).$$

Inserting the expressions (41) and (40) and applying the rules in Lemma 2.2, yields

$$\begin{aligned}
A_3A_2 &= T(A,B)W(A)T(A,B)V(A) + T(A,B)W(A)BT(A,B)W(A)\\
&= T(A,B)T(A,B)W(-A)V(A) - T(A,B)W(A)T(A,B)BW(A)\\
&= W(-A)V(A) - T(A,B)T(A,B)W(-A)BW(A)\\
&= W(-A)V(A) - [BW(-A) - W'(-A)]W(A)\\
&= W(-A)V(A) - BW(-A)W(A) + W'(-A)W(A),
\end{aligned}$$

and hence, we obtain

$$A_2A_3 + A_3A_2 = V(-A)W(A) + W(-A)V(A) + W'(-A)W(A) = 0.$$

We shall now consider the functional-differential equation

$$V(-A)W(A) + W(-A)V(A) + W'(-A)W(A) = 0. \tag{43}$$

From $AB - BA = I$ it follows that $(-A)(-B) - (-B)(-A) = I$, and hence together with $\tilde{A}_1 = -A$ the expressions

$$\begin{aligned}
\tilde{A}_2 &= T(-A,-B)V(-A) - BT(-A,-B)W(-A),\\
\tilde{A}_3 &= T(-A,-B)W(-A),
\end{aligned}$$

will satisfy the same commutation relations as (40) and (41). Thus, it follows that

$$\tilde{A}_2\tilde{A}_3 + \tilde{A}_3\tilde{A}_2 = V(A)W(-A) + W(A)V(-A) + W'(A)W(-A) = 0,$$

and we also have the equation

$$V(A)W(-A) + W(A)V(-A) + W'(A)W(-A) = 0, \tag{44}$$

We shall look for the general solution to (43) and (44) in the form of formal power series

$$V(A) = \sum_{l=0}^{\infty} v_l A^l, \qquad W(A) = \sum_{l=0}^{\infty} w_l A^l, \qquad v_l, w_l \in \mathbb{C}.$$

Subtracting (43) from (44) yields the equation

$$W'(A)W(-A) - W'(-A)W(A) = 0. \tag{45}$$

Integrating (45) and noting that $W(0) = w_0$, we have

$$W(A)W(-A) = w_0^2. \tag{46}$$

Considering first the case when $w_0 = 0$, one has, by substitution of the series $W(A)$ into (46), the infinite system of equations

$$\sum_{l=0}^{s}(-1)^l w_l w_{s-l} = 0, \qquad s = 0, 1, 2, \ldots,$$

having the unique solution $w_0 = w_1 = w_2 = \ldots = 0$, so that $W(A) = 0$. Putting $W(A) = 0$ in the differential equation (43), we see that there is no equation left for $V(A)$, i.e. $V(A)$ can be chosen arbitrarily in the expression (40) for A_2, proving the first statement of the theorem.

Assuming that $w_0 \neq 0$, we can divide both sides of equation (46) by the non-zero constant w_0^2, obtaining the simple equation $g(A)g(-A) = 1$, where $g(A) := W(A)/w_0$. Taking the logarithm of both sides, it follows that $\log g(A)$ has to be an odd power series expression since $\log g(A) + \log g(-A) = 0$. Let $\varphi(A) = \log g(A)$ and we have $W(A) = w_0 g(t) = w_0 \exp(\varphi(A))$, which is the general solution to (46), $\varphi(A)$ being any odd formal power series with complex coefficients. Substituting for $W(A)$ into equation (44) yields

$$V(A)w_0 \exp(\varphi(-A)) + w_0 \exp(\varphi(A))V(-A) + w_0^2\varphi'(A) = 0.$$

This can be written as

$$V(A)[\cosh\varphi(A) - \sinh\varphi(A)] + V(-A)[\cosh\varphi(A) + \sinh\varphi(A)] + w_0\varphi'(A) = 0,$$

$$[V(A) + V(-A)]\cosh\varphi(A) = [V(A) - V(-A)]\sinh\varphi(A) - w_0\varphi'(A),$$

and we have

$$2V(A)\cosh\varphi(A) = [V(A) - V(-A)]\exp(\varphi(A)) - w_0\varphi'(A).$$

Denoting the odd part of $V(A)$ by $V_1(A)$, we obtain

$$V(A)\cosh\varphi(A) = V_1(A)\exp(\varphi(A)) - \tfrac{w_0}{2}\varphi'(A). \tag{47}$$

Let $E(X)$ denote the formal power series defined by

$$E(X) := \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} X^{2n}, \quad E_k = \text{Euler numbers},$$

being the inverse of the formal power series given by $\cosh(X)$, in the sense that $E(X)\cosh(X) = \cosh(X)E(X) = 1$. Multiplying both sides of equation (47) by $E(\varphi(A))$ yields

$$V(A) = E(\varphi(A))V_1(A)\exp(\varphi(A)) - \tfrac{w_0}{2}E(\varphi(A))\,\varphi'(A).$$

We have an expression for $V(A)$ in terms of the odd power series $\varphi(A)$ and the odd part $V_1(A)$ of $V(A)$. $V_1(A)$ can be chosen arbitrarily from the set of formal odd power series with coefficients from $\mathbb{C}$. Writing this as $V_1(A) = w_0\psi(A)$ with w_0 and $\psi(A)$ arbitrary, we have

$$V(A) = w_0 E(\varphi(A))\exp(\varphi(A))\,\psi(A) - \tfrac{w_0}{2}E(\varphi(A))\,\varphi'(A). \tag{48}$$

where $\varphi(A)$ and $\psi(A)$ are arbitrary odd formal power series with complex coefficients. ∎

The simple expressions obtained for the combinations A_2A_3 and A_3^2 in Corollary 2.4, being essentially pure series in A, can now be given in a more explicit form. We formulate the following:

Corollary 2.6 *Suppose A and B are two elements in some complex associative algebra with unity I satisfying the Heisenberg canonical commutation relation $AB - BA = I$. Put $A_1 = A$ and let A_2 and A_3 be formal power series in A and B in the (B, A)-normal form given as*

$$A_2 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}B^jA^k, \quad A_3 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \tilde{a}_{jk}B^jA^k,$$

where the coefficients $a_{jk}, \tilde{a}_{jk} \in \mathbb{C}$. If A_1, A_2 and A_3 satisfy the commutation relations

$$A_1A_2 + A_2A_1 = A_3, \quad A_1A_3 + A_3A_1 = 0, \quad A_2A_3 + A_3A_2 = 0,$$

then we have

$$\begin{aligned} A_2A_3 &= c\,B - c\,E(\varphi(A))[\psi(A) + \tfrac{1}{2}e^{\varphi(A)}\varphi'(A)], \\ A_3^2 &= c\,I, \end{aligned}$$

where $c \in \mathbb{C}$, $E(t)$ is the exponential generating function of the Euler numbers E_k given by

$$E(t) = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} t^{2n},$$

and both $\varphi(A)$ and $\psi(A)$ are odd formal power series expressions in A with complex coefficients.

Proof. By Corollary 2.4 we know that

$$\begin{aligned} A_2 A_3 &= V(-A)W(A) + BW(-A)W(A), \\ A_3^2 &= W(-A)W(A), \end{aligned}$$

where in the present case the formal power series $V(A)$ and $W(A)$, appearing in the expressions for A_2 and A_3 in Theorem 2.1, must be chosen such that the third relation $A_2A_3 + A_3A_2 = 0$ is satisfied. The general solution to that problem is found in the proof of Theorem 2.5 and is of the form

$$\begin{aligned} V(A) &= w_0 E(\varphi(A)) \exp(\varphi(A))\, \psi(A) - \tfrac{1}{2} w_0 E(\varphi(A))\, \varphi'(A), \\ W(A) &= w_0 \exp(\varphi(A)), \end{aligned}$$

where $w_0 \in \mathbb{C}$, $\varphi(A)$ and $\psi(A)$ are arbitrary odd formal power series with coefficients in $\mathbb{C}$ and $E(t) = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} t^{2n}$. This yields

$$\begin{aligned} W(-A)W(A) &= w_0^2 \exp(\varphi(-A)) \exp(\varphi(A)) = w_0^2 \exp(\varphi(A) + \varphi(-A)) \\ &= w_0^2 \exp(\varphi(A) - \varphi(A)) = w_0^2\, I, \\ V(-A)W(A) &= w_0^2 E(\varphi(-A)) \exp(\varphi(-A))\, \psi(-A) \exp(\varphi(A)) \\ &\quad - \tfrac{1}{2} w_0^2 E(\varphi(-A))\, \varphi'(-A) \exp(\varphi(A)) \\ &= -w_0^2 E(\varphi(A)) \psi(A) - \tfrac{1}{2} w_0^2 E(\varphi(A))\, \varphi'(A) \exp(\varphi(A)) \\ &= -w_0^2 E(\varphi(A)) [\psi(A) + \tfrac{1}{2} \exp(\varphi(A))\, \varphi'(A)], \end{aligned}$$

proving the statement of the corollary. ∎

Remark 2.7 Note that exchange of A_1 and A_2 does not change commutation relations (1). So, by exchanging A_1 and A_2 in all statements of the article, we get other expressions for A_1 and A_2 in terms of the Heisenberg generators.

3 Some particular bosonic representations

In this section we will describe some non-trivial particular representations of the color analogue of the Heisenberg Lie algebra defined by the commutation relations (1). All examples are based on the general statement in Theorem 2.5 and correspond to simple specific choices of the odd formal power series $\varphi(A)$ and $\psi(A)$. The constant c is unimportant (except when $c = 0$, see Example 3.1) and we will put c=1 unless stated otherwise.

As a concrete example of elements satisfying the Heisenberg commutation relation (2), we can consider the differentiation and multiplication operators ∂_x and M_x defined on the linear space $\mathbb{C}[x]$, consisting of all complex-valued polynomial functions of a single real variable x. If $f(x) = \sum_{k=0}^{n} f_k x^k$, then by definition

$$(\partial_x f)(x) = \sum_{k=1}^{n} f_k k x^{k-1}, \qquad (M_x f)(x) = x f(x)$$

and we have the well-known relation $\partial_x M_x - M_x \partial_x = I$. As a basis for $\mathbb{C}[x]$ we can take the set of monomials $\{1, x, x^2, \ldots, x^n, x^{n+1}, \ldots\}$. Acting on an arbitrary basis vector x^n, we find

$$\begin{aligned} T(\partial_x, M_x)(x^n) &= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} M_x^k \partial_x^k x^n = \sum_{k=0}^{n} \frac{(-2)^k}{k!} x^k \frac{n!}{(n-k)!} x^{n-k} \\ &= x^n \sum_{k=0}^{n} (-2)^k \binom{n}{k} = (-1)^n x^n = (-x)^n. \end{aligned}$$

In fact, if g is an analytic function on $\mathbb{R}$, we have by Taylor's Theorem

$$e^{\partial_x} g(x) = \sum_{n=0}^{\infty} \frac{\partial_x^n}{n!} g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(x)}{n!} = g(x+1), \tag{49}$$

$$\begin{aligned} T(\partial_x, M_x) g(x) &= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} M_x^k \partial_x^k g(x) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} x^k g^{(k)}(x) \\ &= \sum_{k=0}^{\infty} \frac{g^{(k)}(x)}{k!} (-2x)^k = g(x - 2x) = g(-x). \end{aligned} \tag{50}$$

Example 3.1 By Theorem 2.5 we have a solution corresponding to $A_3 = 0$ given by

$$A_1 = A, \quad A_2 = T(A, B) V(A), \quad A_3 = 0,$$

where $V(A)$ can be any power series in A having complex coefficients. In this case, there is only one non-trivial relation to satisfy. Applying rule (c) of Lemma 2.2 we readily obtain

$$\begin{aligned} A_1A_2 + A_2A_1 &= AT(A,B)V(A) + T(A,B)V(A)A \\ &= -T(A,B)AV(A) + T(A,B)V(A)A = 0. \end{aligned}$$

For any non-zero $V(A)$, we obviously have A_2 given as an infinite power series expression in A and B. This is to be expected as a consequence of Theorem 2.1 (a). Taking $V(A) = 0$, we obtain the trivial realization $A_1 = A$, and $A_2 = A_3 = 0$. As we have shown in Section 2, this is the only possible solution, when A_2 and A_3 are polynomials in A and B.

Assuming that $V(A) = \sum_{l=0}^{\infty} v_l A^l$, we obtain for our simple solution above

$$\begin{aligned} &A_1 = \partial_x, \\ &A_2 = T(\partial_x, M_x)V(\partial_x) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} M_x^k \partial_x^k \sum_{l=0}^{\infty} v_l \partial_x^l, \\ &A_3 = 0. \end{aligned}$$

Acting on an arbitrary basis vector x^n, we find

$$\begin{aligned} A_1(x^n) &= \partial_x x^n = n x^{n-1}, \\ V(\partial_x)(x^n) &= \sum_{l=0}^{\infty} v_l \partial_x^l x^n = \sum_{l=0}^{n} v_l \frac{n!}{(n-l)!} x^{n-l}, \\ A_2(x^n) &= T(\partial_x, M_x)V(\partial_x)x^n = \sum_{l=0}^{n} v_l\, l! \binom{n}{l} (-x)^{n-l}. \end{aligned}$$

Example 3.2 Choosing $\varphi = \psi \equiv 0$ in the general solution expressed in Theorem 2.5, we obtain

$$A_1 = A, \quad A_2 = BT(A,B), \quad A_3 = T(A,B)$$

so in this case we have the simple relation $A_2 = BA_3$. Considering the same situation as in Example 3.1 with $A = \partial_x$ and $B = M_x$ defined on the linear

space $\mathbb{C}[x]$, we have here

$$\begin{aligned} A_1 &= \partial_x, \\ A_2 &= M_x T(\partial_x, M_x) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} M_x^{k+1} \partial_x^k, \\ A_3 &= T(\partial_x, M_x) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} M_x^k \partial_x^k. \end{aligned}$$

These operators are defined on the whole of $\mathbb{C}[x]$, and by Theorem 2.5 they satisfy relations (1). Acting on an arbitrary basis vector x^n, we obtain

$$\begin{aligned} A_1(x^n) &= \partial_x x^n = n x^{n-1}, \\ A_2(x^n) &= M_x T(\partial_x, M_x) x^n = M_x(-x)^n = (-1)^n x^{n+1}, \\ A_3(x^n) &= T(\partial_x, M_x) x^n = (-x)^n. \end{aligned}$$

So, for any polynomial $p(x) \in \mathbb{C}[x]$, we have

$$(A_1 p)(x) = p'(x), \quad (A_2 p)(x) = x p(-x), \quad (A_3 p)(x) = p(-x).$$

These three operators can be defined for any differentiable function f and they satisfy the commutation relations (1), as proved by the following simple calculations:

$$\begin{aligned} A_1 A_2 f(x) &= \partial_x (x f(-x)) = f(-x) - x f'(-x), \\ A_2 A_1 f(x) &= A_2 f'(x) = x f'(-x), \\ A_1 A_3 f(x) &= \partial_x f(-x) = -f'(-x), \\ A_3 A_1 f(x) &= A_3 f'(x) = f'(-x), \\ A_2 A_3 f(x) &= A_2 f(-x) = x f(x), \\ A_3 A_2 f(x) &= A_3 x f(-x) = -x f(x). \end{aligned}$$

So,

$$\begin{aligned} (A_1 A_2 + A_2 A_1) f(x) &= f(-x) = A_3 f(x), \\ (A_1 A_3 + A_3 A_1) f(x) &= -f'(-x) + f'(-x) = 0, \\ (A_2 A_3 + A_3 A_2) f(x) &= x f(x) - x f(x) = 0. \end{aligned}$$

Example 3.3 Let $C^\infty(\mathbb{R})$ be the set of all complex-valued infinitely differentiable functions on the real line. An arbitrary function f can be written in a unique way as a sum $f(x) = f_0(x) + f_1(x)$ of its even and odd parts, where

$$f_0(x) = \frac{f(x) + f(-x)}{2}, \qquad f_1(x) = \frac{f(x) - f(-x)}{2}.$$

If $C_0^\infty(\mathbb{R})$ and $C_1^\infty(\mathbb{R})$ denote the subsets of $C^\infty(\mathbb{R})$ consisting of even and odd infinitely differentiable functions respectively, then this means that $C^\infty(\mathbb{R})$ can be expressed as a direct sum $C^\infty(\mathbb{R}) = C_0^\infty(\mathbb{R}) \oplus C_1^\infty(\mathbb{R})$. Now, let A_1, A_2 and A_3 be defined on $C^\infty(\mathbb{R})$, as in the previous example, by the equations

$$(A_1 f)(x) = f'(x), \quad (A_2 f)(x) = x f(-x), \quad (A_3 f)(x) = f(-x).$$

By considering different domains of definition for these operators by restricting to the subspaces considered above, we have

$$\begin{aligned}
&A_1 : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad A_1 : C_0^\infty(\mathbb{R}) \to C_1^\infty(\mathbb{R}), \quad A_1 : C_1^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}),\\
&A_2 : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad A_2 : C_0^\infty(\mathbb{R}) \to C_1^\infty(\mathbb{R}), \quad A_2 : C_1^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}),\\
&A_3 : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad A_3 : C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}), \quad A_3 : C_1^\infty(\mathbb{R}) \to C_1^\infty(\mathbb{R}).
\end{aligned}$$

where $A_2 = M_x$ on $C_0^\infty(\mathbb{R})$, $A_2 = -M_x$ on $C_1^\infty(\mathbb{R})$, $A_3 = I$ on $C_0^\infty(\mathbb{R})$, and $A_3 = -I$ on $C_1^\infty(\mathbb{R})$. Now, define

$$\mathbf{A}_i : C_0^\infty(\mathbb{R}) \oplus C_1^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}) \oplus C_1^\infty(\mathbb{R}), \; i = 1, 2, 3$$

where the operators $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are given by the operator matrices

$$\mathbf{A}_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & -M_x \\ M_x & 0 \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Using the relation $\partial_x M_x - M_x \partial_x = I$, one easily verifies by direct matrix multiplication that the operators $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ satisfy the relations

$$\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1 = \mathbf{A}_3, \quad \mathbf{A}_1\mathbf{A}_3 + \mathbf{A}_3\mathbf{A}_1 = \mathbf{0}, \quad \mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_3\mathbf{A}_2 = \mathbf{0},$$

where $\mathbf{0}$ denotes the 2×2 zero matrix.

Actually, if we let $\mathbb{C}^{\mathbb{R}}$ and $D(\mathbb{R})$ denote the sets of all complex-valued functions on $\mathbb{R}$ and all complex-valued differentiable functions on $\mathbb{R}$ respectively, then clearly $\mathbf{A}_1$ can be defined on the space $D(\mathbb{R}) = D_0(\mathbb{R}) \oplus D_1(\mathbb{R})$ while $\mathbf{A}_2$ and $\mathbf{A}_3$ are well-defined on the whole of $\mathbb{C}^{\mathbb{R}} = \mathbb{C}_0^{\mathbb{R}} \oplus \mathbb{C}_1^{\mathbb{R}}$. Here

subscripts 0 and 1 have the same meaning as above, i.e. indicate subsets of even and odd functions. It follows that the three relations will be satisfied if the domain of definition is chosen as $D(\mathbb{R}) = D_0(\mathbb{R}) \oplus D_1(\mathbb{R})$.

Example 3.4 The representations of (1) described in Example 3.3 can be generalized as follows. Let H be a linear space and H_0 and H_1 be subspaces of H such that $H_0 \cap H_1 = \{0\}$. Consider $H_0 \oplus H_1$, the subspace of H which is a direct sum of H_0 and H_1. Any linear operator Y on $H_0 \oplus H_1$ can be written as an operator matrix $Y = \left(\begin{smallmatrix} Y_{00} & Y_{01} \\ Y_{10} & Y_{11} \end{smallmatrix}\right)$, where $Y_{jk} : H_k \to H_j$ for $j,k \in \{0,1\}$ are linear operators. Suppose A and B are linear operators on H satisfying on H the Heisenberg canonical commutation relation $AB - BA = I$. Then the linear operators

$$A_1 = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

on $H_0 \oplus H_1$ satisfy the commutation relations (1) of the graded analogue of the Heisenberg Lie algebra. This can be proved by the following calculations:

$$\begin{aligned}
A_1A_2 + A_2A_1 &= \begin{pmatrix} AB & 0 \\ 0 & -AB \end{pmatrix} + \begin{pmatrix} -BA & 0 \\ 0 & BA \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = A_3, \\
A_1A_3 + A_3A_1 &= \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix} + \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0, \\
A_2A_3 + A_3A_2 &= \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} + \begin{pmatrix} 0 & -B \\ -B & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.
\end{aligned}$$

Another way to form such block representations is to use the tensor product. For some linear space H we consider the tensor product $\mathbb{C}^2 \otimes H$, being also a linear space over $\mathbb{C}$. Let A and B be operators on H satisfying $AB - BA = I$. Then the operators

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes A, \quad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes B, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I$$

on $\mathbb{C}^2 \otimes H$ satisfy relations (1). Direct computation, using the rules for the

tensor product, shows that

$$\begin{aligned} A_1A_2 + A_2A_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes AB + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes BA \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes (AB - BA) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I = A_3, \\ A_1A_3 + A_3A_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes A + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes A = 0, \\ A_2A_3 + A_3A_2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes B + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes B = 0. \end{aligned}$$

Introducing the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we can write $A_1 = \sigma_1 \otimes A$, $A_2 = -i\sigma_2 \otimes B$, and $A_3 = \sigma_3 \otimes I$. Among familiar simple properties of the Pauli matrices, we have that $\sigma_1\sigma_2 = i\sigma_3$ and any two different Pauli matrices anticommute. Using these relations, it follows immediately that A_1, A_2 and A_3 must satisfy (1).

Example 3.5 Let s be a positive odd integer and take $\varphi(A) = 0$ and $\psi(A) = A^s$ in the general solution given by Theorem 2.5, yielding

$$A_1 = A, \quad A_2 = T(A,B)A^s + BT(A,B), \quad A_3 = T(A,B).$$

Considering as in Example 3.1 the case when $A = \partial_x$ and $B = M_x$ defined on the linear space $\mathbb{C}[x]$, we have here

$$\begin{aligned} &A_1 = \partial_x, \\ &A_2 = T(\partial_x, M_x)\partial_x^s + M_x T(\partial_x, M_x) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!}(M_x^k \partial_x^{k+s} + M_x^{k+1}\partial_x^k), \\ &A_3 = T(\partial_x, M_x) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} M_x^k \partial_x^k, \end{aligned}$$

satisfying (1) on $\mathbb{C}[x]$ by Theorem 2.5. In a similar way as in Example 3.2, we can define A_1, A_2 and A_3 on the space $D^{s+1}(\mathbb{R})$ of all $s+1$ times differentiable functions on the real line, by the equations

$$(A_1 f)(x) = f'(x), \quad (A_2 f)(x) = f^{(s)}(-x) + xf(-x), \quad (A_3 f)(x) = f(-x).$$

By direct computation we have

$$\begin{aligned}
A_1A_2f(x) &= -f^{(s+1)}(-x) + f(-x) - xf'(-x),\\
A_2A_1f(x) &= f^{(s+1)}(-x) + xf'(-x),\\
(A_1A_2 + A_2A_1)f(x) &= f(-x) = A_3f(x),\\
(A_1A_3 + A_3A_1)f(x) &= -f'(-x) + f'(-x) = 0,\\
A_2A_3f(x) &= (-1)^s f^{(s)}(x) + xf(x),\\
A_3A_2f(x) &= f^{(s)}(x) - xf(x),\\
(A_2A_3 + A_3A_2)f(x) &= (1 + (-1)^s)f^{(s)}(x) = 0.
\end{aligned}$$

Note that for even values of s, $s = 2r$, $r \in \mathbb{N}$, we get operators A_1, A_2 and A_3 satisfying the commutation relations

$$A_1A_2 + A_2A_1 = A_3, \quad A_1A_3 + A_3A_1 = 0, \quad A_2A_3 + A_3A_2 = 2A_1^{2r}.$$

These relations do not correspond to any color Lie algebra, though.

As in Example 3.3, let $C_0^\infty(\mathbb{R})$ and $C_1^\infty(\mathbb{R})$ be the subspaces of $C^\infty(\mathbb{R})$ consisting of even and odd infinitely differentiable functions respectively. Then we have $C^\infty(\mathbb{R}) = C_0^\infty(\mathbb{R}) \oplus C_1^\infty(\mathbb{R})$, and

$$A_2 : C_0^\infty(\mathbb{R}) \to C_1^\infty(\mathbb{R}), \qquad A_2 : C_1^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}),$$

where $A_2 = M_x - \partial_x^s$ on $C_0^\infty(\mathbb{R})$ and $A_2 = \partial_x^s - M_x$ on $C_1^\infty(\mathbb{R})$. Hence, we can define

$$\mathbf{A}_i : C_0^\infty(\mathbb{R}) \oplus C_1^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}) \oplus C_1^\infty(\mathbb{R}),\ i = 1, 2, 3$$

where the operators $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are given by the operator matrices

$$\mathbf{A}_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & \partial_x^s - M_x \\ M_x - \partial_x^s & 0 \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Since, for any non-negative s we have $\partial_x(M_x - \partial_x^s) - (M_x - \partial_x^s)\partial_x = I$, it follows from the general result in Example 3.4 that the operators $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ satisfy the relations

$$\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1 = \mathbf{A}_3, \quad \mathbf{A}_1\mathbf{A}_3 + \mathbf{A}_3\mathbf{A}_1 = \mathbf{0}, \quad \mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_3\mathbf{A}_2 = \mathbf{0},$$

where $\mathbf{0}$ denotes the 2×2 zero matrix. Using the notation introduced in Example 3.3 and above, we have that $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ can be defined on

$D(\mathbb{R}) = D_0(\mathbb{R}) \oplus D_1(\mathbb{R})$, $D^s(\mathbb{R}) = D_0^s(\mathbb{R}) \oplus D_1^s(\mathbb{R})$ and $\mathbb{C}^{\mathbb{R}} = \mathbb{C}_0^{\mathbb{R}} \oplus \mathbb{C}_1^{\mathbb{R}}$ respectively. For the three relations to be satisfied, it suffices to take $D^{s+1}(\mathbb{R}) = D_0^{s+1}(\mathbb{R}) \oplus D_1^{s+1}(\mathbb{R})$ as the domain of definition for all three operators.

Example 3.6 Let $\varphi(A) = A$ and $\psi(A) = 0$ in the general solution given by Theorem 2.5, yielding

$$A_1 = A, \quad A_2 = BT(A,B)e^A - \tfrac{1}{2}T(A,B)E(A), \quad A_3 = T(A,B)e^A.$$

Considering once again as in Example 3.1 the operators $A = \partial_x$ and $B = M_x$ defined on the linear space $\mathbb{C}[x]$, we have

$$\begin{aligned} A_1 &= \partial_x, \\ A_2 &= M_x T(\partial_x, M_x)e^{\partial_x} - \tfrac{1}{2}T(\partial_x, M_x)E(\partial_x), \\ A_3 &= T(\partial_x, M_x)e^{\partial_x}, \end{aligned}$$

or explicitly in the form of operator power series

$$\begin{aligned} A_1 &= \partial_x, \\ A_2 &= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} M_x^{k+1}\partial_x^k \sum_{n=0}^{\infty} \frac{\partial_x^n}{n!} - \tfrac{1}{2}\sum_{k=0}^{\infty} \frac{(-2)^k}{k!} M_x^k \partial_x^k \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!}\partial_x^{2n}, \\ A_3 &= \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} M_x^k \partial_x^k \sum_{n=0}^{\infty} \frac{\partial_x^n}{n!}. \end{aligned}$$

These operators are defined on the whole of the polynomial space $\mathbb{C}[x]$, and by Theorem 2.5 they satisfy (1) on $\mathbb{C}[x]$. For any differentiable function f, we can now define A_1, A_2 and A_3 by the equations

$$\begin{aligned} (A_1 f)(x) &= f'(x), \\ (A_2 f)(x) &= x f(1-x) - \tfrac{1}{2}\sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} f^{(2n)}(-x), \\ (A_3 f)(x) &= f(1-x). \end{aligned}$$

By direct computation we have

$$\begin{aligned} (A_1A_2 + A_2A_1)f(x) = f(1-x) - xf'(1-x) + \tfrac{1}{2}\sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} f^{(2n+1)}(-x) \\ + xf'(1-x) - \tfrac{1}{2}\sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} f^{(2n+1)}(-x) = f(1-x) = A_3 f(x). \end{aligned}$$

$$(A_1A_3 + A_3A_1)f(x) = -f'(1-x) + f'(1-x) = 0,$$

$$\begin{aligned}(A_2A_3 + A_3A_2)f(x) &= xf(1-(1-x)) - \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}f^{(2n)}(1+x) \\ &\quad + (1-x)f(1-(1-x)) - \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}f^{(2n)}(-(1-x)) \\ &= f(x) - \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}[f^{(2n)}(x-1) + f^{(2n)}(x+1)].\end{aligned}$$

Thus, the relation $(A_2A_3 + A_3A_2)f(x) = 0$ is satisfied if and only if the function f satisfies

$$f(x) = \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}[f^{(2n)}(x-1) + f^{(2n)}(x+1)]. \tag{51}$$

The relations (1) hold on $\mathbb{C}[x]$ and so (51) holds for $f \in \mathbb{C}[x]$. We now give an independent combinatorial proof, showing that this formula holds when $f(x)$ is a polynomial in $\mathbb{C}[x]$. Let $f(x) = x^n$, where n is a non-negative integer, and consider the sum

$$\begin{aligned}s_n &= \sum_{k=0}^{\infty}\frac{E_{2k}}{(2k)!}[f^{(2k)}(x-1) + f^{(2k)}(x+1)] \\ &= \sum_{k=0}^{[\frac{n}{2}]} E_{2k}\binom{n}{2k}[\,(x-1)^{n-2k} + (x+1)^{n-2k}\,] \\ &= \sum_{k=0}^{[\frac{n}{2}]} E_{2k}\binom{n}{2k}\left[\sum_{\nu=0}^{n-2k}\binom{n-2k}{\nu}x^{\nu}(-1)^{n-2k-\nu} + \sum_{\nu=0}^{n-2k}\binom{n-2k}{\nu}x^{\nu}\right] \\ &= \sum_{k=0}^{[\frac{n}{2}]} E_{2k}\binom{n}{2k}\sum_{\nu=0}^{n-2k}\binom{n-2k}{\nu}\left[\,1 + (-1)^{n-2k-\nu}\,\right]x^{\nu}.\end{aligned}$$

Since the Euler numbers $E_{2k+1} = 0$ for all non-negative integer values k (see

Appendix B), we can write this as

$$
\begin{aligned}
s_n &= \sum_{m=0}^{n} E_m \binom{n}{m} \sum_{\nu=0}^{n-m} \binom{n-m}{\nu} \left[\, 1 + (-1)^{n-m-\nu} \,\right] x^\nu \\
&= \sum_{\nu=0}^{n} \sum_{m=0}^{n-\nu} E_m \binom{n}{m} \binom{n-m}{\nu} \left[\, 1 + (-1)^{n-m-\nu} \,\right] x^\nu
\end{aligned}
$$

We have s_n expressed as a polynomial of degree n and with coefficients $p_{n\nu}$ given by

$$
\begin{aligned}
p_{n\nu} &= \sum_{m=0}^{n-\nu} E_m \binom{n}{m} \binom{n-m}{\nu} \left[\, 1 + (-1)^{n-m-\nu} \,\right] \\
&= \frac{n!}{\nu!} \sum_{m=0}^{n-\nu} \frac{E_m}{m!(n-\nu-m)!} \left[\, 1 + (-1)^{n-\nu-m} \,\right] \\
&= \frac{n!}{\nu!} \sum_{k=0}^{\left[\frac{n-\nu}{2}\right]} \frac{E_{2k}}{(2k)!\,(n-\nu-2k)!} \left[\, 1 + (-1)^{n-\nu-2k} \,\right].
\end{aligned}
$$

For odd values of $n - \nu$, we observe that $p_{n\nu} = 0$ because of the factor $1 + (-1)^{n-\nu-2k}$ being zero, while for even $n - \nu$ we can write $n - \nu = 2r$ for $r \in \mathbb{N}$, and hence

$$
\begin{aligned}
p_{n\nu} &= 2\,\frac{n!}{\nu!} \sum_{k=0}^{r} \frac{E_{2k}}{(2k)!\,(2r-2k)!} = \frac{2\,n!}{(2r)!\,\nu!} \sum_{k=0}^{r} E_{2k} \binom{2r}{2k} \\
&= \frac{2\,n!}{(2r)!\,\nu!} \left(\frac{2^{2r+1}}{2r+1} (2^{2r+1} - 1) B_{2r+1} + 2\delta_{0,2r} \right),
\end{aligned}
$$

where B_n are the Bernoulli numbers and $\delta_{k,l}$ is Kronecker's delta. Here we have used formula (5.1.3.2) on page 385 in [23]. Some useful facts about the Bernoulli numbers can also be found in Appendix B. For $\nu = n$ we have $r = 0$ and

$$
p_{nn} = 2[2(2-1)B_1 + 2\delta_{00}] = 4(B_1 + \delta_{00}) = 4(-\tfrac{1}{2} + 1) = 2.
$$

Since $B_{2r+1} = 0$ for all $r \geq 1$, we have shown that $p_{n\nu} = 0$ when $\nu < n$, and hence

$$
s_n = \sum_{\nu=0}^{n} p_{n\nu} x^\nu = p_{nn} x^n = 2x^n.
$$

For every non-negative integer n, it holds that

$$x^n = \tfrac{1}{2} \sum_{k=0}^{[\frac{n}{2}]} \frac{E_{2k}}{(2k)!} \left(\frac{n!}{(n-2k)!} (x-1)^{n-2k} + \frac{n!}{(n-2k)!} (x+1)^{n-2k} \right).$$

It follows, that for every polynomial $p(x) \in \mathbb{C}[x]$, we obtain

$$p(x) = \tfrac{1}{2} \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} [p^{(2k)}(x-1) + p^{(2k)}(x+1)].$$

Another way to deduce this is to proceed by operator methods. Let us define the shift operator S and the differentiation operator D as follows: For any function f defined on the real line $\mathbb{R}$, the action of S on f is given by $(Sf)(x) = f(x+1)$. The inverse S^{-1} exists and acts as $(S^{-1}f)(x) = f(x-1)$. The operator D is defined on $D(\mathbb{R})$, the class of all differentiable functions on the real line, as $Df = f'$. If we restrict the domain of definition of both S and D to the space of analytic functions on $\mathbb{R}$ then, as proved in (49), we have the equality $S = \exp(D)$. Moreover, it holds that

$$I + e^{2D} = I + S^2 = (S^{-1} + S)S = (S^{-1} + S)e^D.$$

Using the exponential generating function of the Euler numbers and restricting the domain of definition further to analytic functions f on $\mathbb{R}$, such that

$$2e^D (I + e^{2D})^{-1} f = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} D^{2k} f$$

holds (and in particular all parts of the equality exist), then we have

$$\begin{aligned} I &= (S^{-1} + S)e^D (I + e^{2D})^{-1} = \tfrac{1}{2}(S^{-1} + S) \left(2e^D (I + e^{2D})^{-1}\right) \\ &= \tfrac{1}{2}(S^{-1} + S) \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} D^{2k} = \tfrac{1}{2} \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} (S^{-1} + S) D^{2k}. \end{aligned}$$

We have obtained

$$I = \tfrac{1}{2} \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} (S^{-1} D^{2k} + S D^{2k}),$$

or by acting on a polynomial function $p(x)$

$$p(x) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{E_{2k}}{(2k)!}[p^{(2k)}(x-1)+p^{(2k)}(x+1)].$$

The interpolation formula (51) can be shown to hold for a larger class of functions than just $\mathbb{C}[x]$, but not for all analytic functions on $\mathbb{R}$. An illustrative example is the exponential function e^{ax}, where the formula can be shown to hold for $|a| < \pi/2$ (cf. Appendix B). In order to extend the domain of definition for A_1, A_2 and A_3 from $\mathbb{C}[x]$ to a bigger space, it would be interesting to characterize this class of functions and to see how the operator A_2 can be defined on a larger domain than is done here.

Example 3.7 This example contains Examples 3.1, 3.2, 3.5 and 3.6 for specific values of the parameters defining φ and ψ. Let s be a positive odd integer and take $\varphi(A) = \alpha A$ and $\psi(A) = \beta_s A^s$ in the general solution given by Theorem 2.5. Then we obtain

$$\begin{aligned}
&A_1 = A, \quad A_3 = c\,T(A,B)e^{\alpha A},\\
&A_2 = c\,T(A,B)E(\alpha A)[e^{\alpha A}\beta_s A^s - \tfrac{1}{2}\alpha] + c\,BT(A,B)e^{\alpha A}
\end{aligned}$$

Considering once again as in Example 3.1 the operators $A = \partial_x$ and $B = M_x$ defined on the linear space $\mathbb{C}[x]$, we have

$$\begin{aligned}
&A_1 = \partial_x,\\
&A_2 = c\,T(\partial_x, M_x)E(\alpha\partial_x)[e^{\alpha\partial_x}\beta_s\partial_x^s - \tfrac{1}{2}\alpha] + c\,M_x T(\partial_x, M_x)e^{\alpha\partial_x},\\
&A_3 = c\,T(\partial_x, M_x)e^{\alpha\partial_x}.
\end{aligned}$$

We can now define A_1, A_2 and A_3 on the polynomial space $\mathbb{C}[x]$ by the

equations

$$
\begin{aligned}
(A_1f)(x) &= f'(x),\\
(A_2f)(x) &= c\,T(\partial_x, M_x)E(\alpha\partial_x)[\beta_s f^{(s)}(x+\alpha) - \tfrac{1}{2}\alpha f(x)] + cxf(-x+\alpha)\\
&= c\,T(\partial_x, M_x)\sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}\alpha^{2n}[\beta_s f^{(s+2n)}(x+\alpha) - \tfrac{1}{2}\alpha f^{(2n)}(x)]\\
&\quad + cxf(-x+\alpha)\\
&= c\sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}\alpha^{2n}[\beta_s f^{(s+2n)}(-x+\alpha) - \tfrac{1}{2}\alpha f^{(2n)}(-x)]\\
&\quad + cxf(-x+\alpha)\\
(A_3f)(x) &= c\,T(\partial_x, M_x)e^{\alpha\partial_x}f(x) = cf(-x+\alpha).
\end{aligned}
$$

By direct computation we have

$$
\begin{aligned}
(A_1A_2f)(x) = c\sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}\alpha^{2n}[-\beta_s f^{(s+2n+1)}(-x+\alpha) + \tfrac{1}{2}\alpha f^{(2n+1)}(-x)]\\
- cxf'(-x+\alpha) + cf(-x+\alpha).
\end{aligned}
$$

$$
\begin{aligned}
(A_2A_1f)(x) = c\sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}\alpha^{2n}[\beta_s f^{(s+2n+1)}(-x+\alpha) - \tfrac{1}{2}\alpha f^{(2n+1)}(-x)]\\
+ cxf'(-x+\alpha).
\end{aligned}
$$

$$(A_1A_2 + A_2A_1)f(x) = cf(-x+\alpha) = (A_3f)(x),$$

$$(A_1A_3 + A_3A_1)f(x) = -cf'(-x+\alpha) + cf'(-x+\alpha) = 0,$$

$$
\begin{aligned}
(A_2A_3f)(x) = c\sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}\alpha^{2n}[c\beta_s(-1)^s f^{(s+2n)}(x) - \tfrac{1}{2}c\alpha f^{(2n)}(x-\alpha)]\\
+ c^2xf(x).
\end{aligned}
$$

$$
\begin{aligned}
(A_3A_2f)(x) = c^2 \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} \alpha^{2n} [\beta_s f^{(s+2n)}(x) - \tfrac{1}{2}\alpha f^{(2n)}(x+\alpha)] \\
+ c^2(-x+\alpha)f(x).
\end{aligned}
$$

$$
\begin{aligned}
(A_2A_3 + A_3A_2)f(x) &= c^2 x f(x) - \tfrac{1}{2}c^2 \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} \alpha^{2n+1} f^{(2n)}(x-\alpha) \\
&\quad + c^2(-x+\alpha)f(x) - \tfrac{1}{2}c^2 \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} \alpha^{2n+1} f^{(2n)}(x+\alpha) \\
&= \alpha c^2 f(x) - \tfrac{1}{2}c^2 \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} \alpha^{2n+1} [f^{(2n)}(x-\alpha) + f^{(2n)}(x+\alpha)].
\end{aligned}
$$

Thus, the relation $(A_2A_3 + A_3A_2)f(x) = 0$ is satisfied if and only if the function f satisfies

$$
f(x) = \tfrac{1}{2} \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} \alpha^{2n} [f^{(2n)}(x-\alpha) + f^{(2n)}(x+\alpha)]. \tag{52}
$$

Example 3.8 Let us have a closer look at the operators studied in Example 3.6, namely

$$
\begin{aligned}
A_1 &= \partial_x, \\
A_2 &= M_x T(\partial_x, M_x) e^{\partial_x} - \tfrac{1}{2}\, T(\partial_x, M_x) E(\partial_x), \\
A_3 &= T(\partial_x, M_x) e^{\partial_x},
\end{aligned}
$$

We can express $E(\partial_x)$ in terms of the generating function for the Euler numbers, obtaining

$$
\begin{aligned}
E(\partial_x) &= 2e^{\partial_x}(e^{2\partial_x}+1)^{-1} = e^{\partial_x}(\tfrac{1}{2} + \tfrac{1}{2}e^{2\partial_x})^{-1} = e^{\partial_x}(1 + \tfrac{1}{2}(e^{2\partial_x}-1))^{-1} \\
&= e^{\partial_x} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} (e^{2\partial_x}-1)^k = e^{\partial_x} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} \sum_{l=0}^{k} \binom{k}{l} e^{2(k-l)\partial_x} (-1)^l \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} e^{(2(k-l)+1)\partial_x}
\end{aligned}
$$

By virtue of equations (49) and (50) in Example 3.6, it is now reasonable to define

$$
\begin{aligned}
&(A_1 f)(x) = f'(x),\\
&(A_2 f)(x) = xf(1-x) - \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} \sum_{l=0}^{k} (-1)^l \binom{k}{l} f(2(k-l)+1-x),\\
&(A_3 f)(x) = f(1-x),
\end{aligned}
$$

where f is a polynomial $\mathbb{C}[x]$. In order to verify the commutation relations, we calculate

$$
\begin{aligned}
A_1 A_2 f(x) &= \partial_x \left(xf(1-x) - \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{(-1)^{k+l}}{2^{k+1}} \binom{k}{l} f(2(k-l)+1-x) \right)\\
&= f(1-x) - xf'(1-x) + \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{(-1)^{k+l}}{2^{k+1}} \binom{k}{l} f'(2(k-l)+1-x)
\end{aligned}
$$

$$
A_2 A_1 f(x) = xf'(1-x) + \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{(-1)^{k+l}}{2^{k+1}} \binom{k}{l} f'(2(k-l)+1-x)
$$

We have then

$$
\begin{aligned}
&(A_1 A_2 + A_2 A_1) f(x) = f(1-x) = A_3 f(x),\\
&(A_1 A_3 + A_3 A_1) f(x) = -f'(1-x) + f'(1-x) = 0.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
A_2 A_3 f(x) &= A_2 f(1-x) = xf(x) - \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{(-1)^{k+l}}{2^{k+1}} \binom{k}{l} f(x-2(k-l))\\
A_3 A_2 f(x) &= A_3 \left(xf(1-x) - \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{(-1)^{k+l}}{2^{k+1}} \binom{k}{l} f(2(k-l)+1-x) \right)\\
&= (1-x)f(x) - \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{(-1)^{k+l}}{2^{k+1}} \binom{k}{l} f(x+2(k-l))
\end{aligned}
$$

and hence

$$
\begin{aligned}
(A_2A_3 + A_3A_2)f(x) &= xf(x) + (1-x)f(x) \\
&\quad - \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{(-1)^{k+l}}{2^{k+1}} \binom{k}{l} [f(x-2(k-l)) + f(x+2(k-l))] \\
&= f(x) - \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{(-1)^{k+l}}{2^{k+1}} \binom{k}{l} [f(x-2(k-l)) + f(x+2(k-l))]
\end{aligned}
$$

In order to satisfy the relation $(A_2A_3 + A_3A_2)f(x) = 0$, we must have

$$
f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} \sum_{l=0}^{k} (-1)^l \binom{k}{l} [f(x-2(k-l)) + f(x+2(k-l))]. \tag{53}
$$

We shall now demonstrate that this formula holds if $f(x)$ is an arbitrary polynomial in x. For this purpose we need to use some properties of the Stirling numbers of the second kind, see Chapter 5 in [4]. We take $f(x) = x^n$ with n a non-negative integer and consider the sum

$$
\begin{aligned}
\sigma_n &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} [f(x-2(k-l)) + f(x+2(k-l))] \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} [(x-2(k-l))^n + (x+2(k-l))^n] \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \sum_{\nu=0}^{n} \binom{n}{\nu} x^{n-\nu} 2^\nu (k-l)^\nu (1+(-1)^\nu) \\
&= \sum_{\nu=0}^{n} \binom{n}{\nu} 2^\nu (1+(-1)^\nu) \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} (k-l)^\nu x^{n-\nu} \\
&= \sum_{\nu=0}^{n} \binom{n}{\nu} 2^\nu (1+(-1)^\nu) \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} l^\nu x^{n-\nu} \\
&= \sum_{\nu=0}^{n} \binom{n}{\nu} 2^\nu (1+(-1)^\nu) \sum_{k=0}^{\infty} \frac{(-1)^k k!}{2^k} S(\nu,k) x^{n-\nu},
\end{aligned}
$$

where we have introduced the Stirling numbers of the second kind, given by

$$S(m,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^m, \quad m \geq 1,\ k \geq 0,$$
$$S(0,0) = 1, \quad S(0,k) = 0, \quad k \geq 1.$$

The sum σ_n is now expressed as a polynomial of degree n with coefficients $q_{n\nu}$ given as

$$q_{n\nu} = \sum_{\nu=0}^{n}\binom{n}{\nu}2^\nu(1+(-1)^\nu)\sum_{k=0}^{\nu}\frac{(-1)^k k!}{2^k}S(\nu,k)$$

since for $k > m$, we have by equation (4.2.2.3) on page 608 in [22]

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^m = 0.$$

For odd values of ν, we note that $q_{n\nu} = 0$ due to the factor $(1+(-1)^\nu)$. Now define $f_0 = g_0 = 0$ and let for all positive integers k, m

$$f_k = \frac{(-1)^k k!}{2^k}, \qquad g_m = \sum_{k=0}^{m} S(m,k) f_k.$$

The exponential generating functions $F(t)$ and $G(t)$ corresponding to the sequences (f_k) and (g_m) respectively, are defined as

$$F(t) = \sum_{k=0}^{\infty} f_k \frac{t^k}{k!}, \qquad G(t) = \sum_{m=0}^{\infty} g_m \frac{t^k}{m!}.$$

We find

$$F(t) = \sum_{k=0}^{\infty} f_k \frac{t^k}{k!} = \sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}t^k = \sum_{k=1}^{\infty}\left(\frac{-t}{2}\right)^k = \frac{-t}{2+t}.$$

By Theorem 5.4.2 in [4] it follows that

$$G(t) = F(e^t - 1) = \frac{1-e^t}{1+e^t},$$

and since

$$G(-t) = \frac{1-e^{-t}}{1+e^{-t}} = \frac{e^t - 1}{e^t + 1} = -G(t),$$

$G(t)$ is an odd generating function, and hence for positive even values of ν

$$g_\nu = \sum_{k=0}^{\nu} S(\nu, k) f_k = \sum_{k=0}^{\nu} \frac{(-1)^k k!}{2^k} S(\nu, k) = 0,$$

yielding the result

$$\sigma_n = \sum_{\nu=0}^{n} q_{n\nu} x^{n-\nu} = q_{n0} x^n = 2x^n.$$

This proves that, for any non-negative integer n, we have

$$x^n = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} \sum_{l=0}^{k} (-1)^l \binom{k}{l} [(x - 2(k-l))^n + (x + 2(k-l))^n].$$

It follows, that for every polynomial $q(x) \in \mathbb{C}[x]$, we obtain

$$q(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} \sum_{l=0}^{k} (-1)^l \binom{k}{l} [q(x - 2(k-l)) + q(x + 2(k-l))].$$

The formula (53) holds for other functions than polynomials. An interesting task would be to find a characterization of this class of functions.

Acknowledgements

The authors would like to thank Professor Bernt Lindström for suggesting the main ideas behind the combinatorial argument in Example 3.8. Thanks are also due to Dr. Lars Hellström and Dr. Edwin Langmann for useful comments. The first author is grateful to the Department of Mathematics at the Lund Institute of Technology for hospitality during his visits in Lund.

Appendix A

Recall that a $\mathbb{Z}_2^n$-graded (color) generalized Lie algebra is a $\mathbb{Z}_2^n$-graded linear space

$$X = \bigoplus_{\gamma \in \mathbb{Z}_2^n} X_\gamma$$

with a bilinear multiplication (bracket) $\langle \cdot , \cdot \rangle : X \times X \to X$ obeying:

Grading axiom: $\langle X_\alpha , X_\beta \rangle \subseteq X_{\alpha+\beta}$.

Graded skew-symmetry: $\langle a , b \rangle = -(-1)^{\alpha\cdot\beta} \langle b , a \rangle$.

Generalized Jacobi identity:

$$(-1)^{\alpha\cdot\gamma} \langle a , \langle b , c \rangle \rangle + (-1)^{\gamma\cdot\beta} \langle c , \langle a , b \rangle \rangle + (-1)^{\beta\cdot\alpha} \langle b , \langle c , a \rangle \rangle = 0$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, $\gamma = (\gamma_1, \ldots, \gamma_n)$ in $\mathbb{Z}_2^n$, and $a \in X_\alpha$, $b \in X_\beta$, $c \in X_\gamma$, where $\alpha \cdot \beta = \sum_{i=1}^n \alpha_i \beta_i$ etc., with $\sum$ meaning addition in $\mathbb{Z}_2$. The elements of $\bigcup_{\gamma \in \mathbb{Z}_2^n} X_\gamma$ are called homogeneous.

Any $\mathbb{Z}_2^n$-graded generalized Lie algebra X can be embedded in its universal enveloping algebra $U(X)$ in such a way that, for homogeneous $a \in X_\alpha$ and $b \in X_\beta$, the bracket $\langle \cdot , \cdot \rangle$ becomes a commutator $[a , b] = ab - ba$ when $\alpha \cdot \beta$ is even, or an anticommutator $\{a , b\} = ab + ba$ when $\alpha \cdot \beta$ is odd [28].

Now take X to be a $\mathbb{Z}_2^3$-graded linear space

$$X = X_{(1,1,0)} \oplus X_{(1,0,1)} \oplus X_{(0,1,1)}$$

with the homogeneous basis $A_1 \in X_{(1,1,0)}$, $A_2 \in X_{(1,0,1)}$, $A_3 \in X_{(0,1,1)}$. The homogeneous components graded by the elements of $\mathbb{Z}_2^3$ different from $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ are zero and so are omitted. If the $\mathbb{Z}_2^3$-graded bilinear multiplication $\langle \cdot , \cdot \rangle$ turns X into a $\mathbb{Z}_2^3$-graded generalized Lie algebra, then $\langle A_i , A_i \rangle = 0$, $i = 1, 2, 3$ and

$$\langle A_1 , A_2 \rangle = c_{12} A_3, \quad \langle A_2 , A_3 \rangle = c_{23} A_1, \quad \langle A_3 , A_1 \rangle = c_{31} A_2 \ .$$

When a and b are in different homogeneous subspaces, it follows that $\langle a , b \rangle = \langle b , a \rangle$, whereas $\langle a , b \rangle = -\langle b , a \rangle$ if a and b belong to the same one. Moreover, the generalized Jacobi identity is valid. Now put $c_{12} = 1$, $c_{23} = 0$ and $c_{31} = 0$. The algebra X so defined has as its universal enveloping algebra the color analogue of the Heisenberg Lie algebra.

Appendix B

The Bernoulli polynomials $B_k(x)$ can be defined in terms of their exponential generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!}\,,\quad |t|<2\pi.$$

The four polynomials of lowest degree are

$$B_0(x)=1,\quad B_1(x)=x-\tfrac{1}{2},\quad B_2(x)=x^2-x+\tfrac{1}{6},\quad B_3(x)=x^3-\tfrac{3}{2}x^2+\tfrac{1}{2}x.$$

The Bernoulli numbers B_k are then defined as the values of $B_k(x)$ at the origin, $B_k=B_k(0)$, from which it follows that $B_0=1$, $B_1=-\frac{1}{2}$, and for $k\geq 1$

$$B_{2k+1}=0,\quad B_{2k}=(-1)^{k+1}\frac{2\,(2k)!}{\pi^{2k}(2^{2k}-1)}\sum_{\nu=0}^{\infty}(2\nu+1)^{-2k}.$$

In a similar way, one can define a sequence of polynomials $E_k(x)$, called the Euler polynomials, by specifying their exponential generating function as

$$\frac{2e^{xt}}{e^t+1}=\sum_{k=0}^{\infty}E_k(x)\frac{t^k}{k!}\,,\quad |t|<\pi.$$

The four polynomials of lowest degree are

$$E_0(x)=1,\quad E_1(x)=x-\tfrac{1}{2},\quad E_2(x)=x^2-x,\quad E_3(x)=x^3-\tfrac{3}{2}x^2+\tfrac{1}{4}.$$

The Euler numbers E_k are then defined as the integers $E_k=2^kE_k(\frac{1}{2})$. It follows that $E_0=1$, $E_1=0$, $E_2=-1$, $E_3=0$, and generally for $k\geq 0$

$$E_{2k+1}=0,\quad E_{2k}=(-1)^k\frac{(2k)!\,2^{2k+2}}{\pi^{2k+1}}\sum_{\nu=0}^{\infty}(-1)^{\nu}(2\nu+1)^{-2k-1}.$$

The Euler numbers have an exponential generating function obtained by setting $x=1/2$ and replacing t by $2t$ in the exponential generating function of the Euler polynomials

$$E(t)=\frac{2e^t}{e^{2t}+1}=\sum_{k=0}^{\infty}E_k\frac{t^k}{k!}.$$

References

[1] V.K. Agrawala, *Invariants of generalized Lie algebras,* Hadronic J. **4** (1981), 444–496.

[2] Yu.A. Bahturin, A.A. Mikhalev, V.M. Petrogradsky, M.V. Zaicev, *Infinite Dimensional Lie Superalgebras,* Walter de Gruyter, Berlin, 1992.

[3] R.E. Block, *The Irreducible Representations of the Lie Algebra $\mathfrak{sl}(2)$ and of the Weyl Algebra,* Adv. Math. **39** (1981), 69–110.

[4] P.J. Cameron, *Combinatorics: topics, techniques, algorithms*, Cambridge University Press, 1994.

[5] L. Corwin, Y. Ne'eman, S. Sternberg, *Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry),* Rev. Modern Phys. **47** (1975), 573–603.

[6] H.S. Green, P.D. Jarvis, *Casimir invariants, characteristic identities and Young diagrams for Colour algebras and superalgebras,* J. Math. Phys. **24** (1983), 1681–1687.

[7] L. Hellström, S.D. Silvestrov, *Commuting Elements in q-Deformed Heisenberg Algebras*, World Scientific, 2000, 256 pp.
(ISBN: 981-02-4403-7).

[8] V.G. Kac, *Lie Superalgebras,* Adv. Math. **26** (1977), 8–96.

[9] M.V. Karasev, V.P. Maslov, *Non-Lie permutation relations,* Russian Math. Surveys **45** (1990), 51–98.

[10] R. Kleeman, *Commutation factors on generalized Lie algebras,* J. Math. Phys. **26** (1985), 2405–2412.

[11] A.K. Kwasniewski, *Clifford- and Grassmann-like algebras – Old and new,* J. Math. Phys. **26** (1985), 2234–2238.

[12] A.K. Kwasniewski, *On Graded Lie-like Algebras,* Bulletin de la Société des sciences et des lettres de Łódź **39**, 6 (1989).

[13] H. Ljungqvist, S.D. Silvestrov, *Involutions in three-dimensional coloured Lie algebras,* Research Reports **6**, Dep. Math., Umeå University, (1996), 36 pp.

[14] J. Lukierski, V. Rittenberg, *Color-de Sitter and color-conformal superalgebras,* Phys. Rev. D **18** (1978), 385–389.

[15] W. Marcinek, *Colour extensions of Lie algebras and superalgebras,* Preprint **746**, University of Wrocław (1990).

[16] W. Marcinek, *Generalized Lie algebras and Related Topics,1,2,* Acta Univ. Wratislaviensis (Matematyka, Fizyka, Astronomia) **LV**, 1170 (1991), 3–52.

[17] M.V. Mosolova, *Functions of non-commuting operators that generate a graded Lie algebra,* Mat. Zametki **29** (1981), 34–45.

[18] V.L. Ostrovskyĭ, Yu.S. Samoĭlenko, *Introduction to the Theory of Representations of Finitely Presented $*$-Algebras. I. Representations by bounded operators*, Rev. Math. Math. Phys. **11**, The Gordon and Breach Publ. Group, 1999.

[19] V.L. Ostrovskii, S.D. Silvestrov, *Representations of the real forms of the graded analogue of the Lie algebra* $sl(2,\mathbb{C})$, Ukrain. Mat. Zh. **44** (1992), 1518–1524; (English translation: Ukrainian Math. J. **44** (1993), 1395–1401).

[20] L. Persson, S.D. Silvestrov, P. Strunk, *Central elements of the second order in three-dimensional generalised Lie algebras*, *Czechoslovak Journal of Physics* **47** (1997), 99–106.
(Also published as Research Report **5**, Dep. Math., Umeå University, (1996), 24 pp.)

[21] K.L. Price, *Primeness Criteria for Universal Enveloping Algebras of Lie Color Algebras*, *Journal of Algebra* **235** (2001), 589–607.

[22] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Integrals and Series, Vol. 1: Elementary Functions,* Gordon and Breach Science Publishers, 1990. (Translated from the Russian original edition: Nauka, Moscow, 1986.)

[23] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Integrals and Series, Vol. 3: More Special Functions,* Gordon and Breach Science Publishers, 1990. (Translated from the Russian original edition: Nauka, Moscow, 1986.)

[24] C.R. Putnam, *Commutation Properties of Hilbert Space Operators and Related Topics,* Springer-Verlag, Berlin Heidelberg, 1967.

[25] V. Rittenberg, D. Wyler, *Generalized Superalgebras,* Nucl. Phys. **B139** (1978), 189–202.

[26] L. Råde, B. Westergren, *Mathematics Handbook for Science and Engineering,* Studentlitteratur, Lund, 1995.

[27] Yu.S. Samoilenko, *Spectral Theory of Families of Self-Adjoint Operators,* Kluwer Acad. Publ., Dordrecht, 1991.

[28] M. Scheunert, *Generalized Lie algebras,* J. Math. Phys. **20** (1979), 712–720.

[29] M. Scheunert, *Graded tensor calculus,* J. Math. Phys. **24** (1983), 2658–2670.

[30] M. Scheunert, *Casimir elements of ε-Lie algebras,* J. Math. Phys. **24** (1983), 2671–2680.

[31] G. Sigurdsson, S.D. Silvestrov, *Canonical involutions in three-dimensional generalised Lie algebras, Czechoslovak Journal of Physics,* **50** (2000), 181–186.

[32] S.D. Silvestrov, *Representations of Commutation Relations. A Dynamical Systems Approach*, Hadronic Journal Supplement, **11** (1996), 1–116.

[33] S.D. Silvestrov, *Hilbert space representations of the graded analogue of the Lie algebra of the group of plane motions,* Studia Mathematica **117** (1996), 195–203.

[34] S.D. Silvestrov, *On the classification of 3-dimensional coloured Lie algebras,* in *Quantum Groups and Quantum Spaces*, Banach Center Publications **40** (1997), 159–170.

[35] N. Weaver, *Mathematical Quantization,* (Studies in advanced mathematics), Chapman & Hall/CRC, 2001.

[36] H. Wielandt, *Über die Unbeschränkheit der Schrödingerschen Operatoren der Quanten-mechanik,* Math. Ann. **121** (1949), 21.

[37] A. Wintner, *The unboundedness of quantum-mechanical matrices,* Phys. Rev. **71** (1947), 738–739.